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OF ELEMENTARY PARTICLES

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§ 1. IN some recent work¹ on the algebra of matrices related to elementary particles of higher spin, the following polynomial was found to be of importance:

$$\eta_s(x) = \frac{(x-s)(x-s+1)\cdots(x+s)}{|2s|} \sum_{r=0}^{2s} \binom{2s}{r} \frac{1}{x-s+r}. \quad (1)$$

Here s denotes the value of the spin of the particles, integral or half odd-integral. The object of this note is to derive a recurrence-relation satisfied by this polynomial.

§ 2. Write $y=2x$, $m=2s$. $\eta_s(x) \equiv \phi_m(y)$.

Then

$$\phi_m(y) = \frac{(y-m)(y-m+2)\cdots(y+m-2)(y+m)}{2^m |m|} \times \sum_{r=0}^m \binom{m}{r} \frac{1}{y-m+2r} \quad (2)$$

We write

$$\psi_m(y) = \frac{(y-m)(y-m+2)\cdots(y+m-2)(y+m)}{2^m |m|} \quad (3a)$$

$$I_m(y) = \sum_{r=0}^m \binom{m}{r} \frac{1}{y-m+2r} \quad (3b)$$

so that

$$\phi_m(y) = \psi_m(y) I_m(y) \quad (4)$$

§ 3. Now

$$\begin{aligned} I_m(y) &= \sum_{r=0}^m \binom{m}{r} \frac{1}{y-m+2r} \\ &= \sum_{r=0}^m \binom{m}{r} \int_0^1 u^{y-m+2r-1} du \\ &= \int_0^1 u^{y-m-1} du \cdot \sum_{r=0}^m \binom{m}{r} u^{2r} \\ \text{i.e., } I_m(y) &= \int_0^1 u^{y-m-1} (1+u^2)^m du \end{aligned} \quad (5)$$

The problem is reduced to finding a reduction formula for I_m .

By integration by parts, using the first factor in the integrand for integration, we have

$$\begin{aligned} I_m(y) &= \frac{2m}{y-m} - \frac{2m}{y-m} \int_0^1 u^{y-m+1} (1+u^2)^{m-1} du \\ &= \frac{2^m}{y-m} - \frac{2^m}{y-m} \left[I_m(y) - \int_0^1 u^{y-m-1} (1+u^2)^{m-1} du \right] \end{aligned} \quad (6a)$$

On the other hand, choosing the second factor in the integrand for integration, we get

$$\begin{aligned} I_{m-2}(y) &= \int_0^1 u^{y-m+1} (1+u^2)^{m-2} du \\ &= \frac{1}{2} \left\{ \frac{2^{m-1}}{m-1} - \frac{y-m}{m-1} \int_0^1 u^{y-m-1} (1+u^2)^{m-1} du \right\} \end{aligned} \quad (6b)$$

Combining (6a) and (6b) we find

$$I_m(y) = \frac{2^m}{y-m} - \frac{2m}{y-m} I_m(y) + \frac{4m(m-1)}{(y-m)^2} \left\{ \frac{2^{m-2}}{m-1} - I_{m-2}(y) \right\}$$

or
$$I_m(y) = \frac{2^m y}{y^2 - m^2} - \frac{4m(m-1)}{y^2 - m^2} I_{m-2}(y)$$

Changing m into $m+2$, this may be written in the form

$$I_{m+2}(y) = \frac{2^{m+2} y}{y^2 - (m+2)^2} - \frac{4(m+1)(m+2)}{y^2 - (m+2)^2} I_m(y) \quad (7)$$

§ 4. From (3a) we have

$$\psi_{m+2}(y) = \frac{y^2 - (m+2)^2}{4(m+1)(m+2)} \psi_m(y) \quad (8)$$

§ 5. We thus have

$$\begin{aligned} \phi_{m+2}(y) &= \psi_{m+2}(y) I_{m+2}(y) \\ &= \frac{y^2 - (m+2)^2}{4(m+1)(m+2)} \times \\ &\quad \left\{ \frac{2^{m+2} y}{y^2 - (m+2)^2} - \frac{4(m+1)(m+2)}{y^2 - (m+2)^2} I_m(y) \right\} \psi_m(y) \\ &= \frac{2^m y}{(m+1)(m+2)} \psi_m(y) - \phi_m(y) \end{aligned}$$

$$\therefore \phi_{m+2}(y) + \phi_m(y) = \frac{2^m y}{(m+1)(m+2)} \psi_m(y) \quad (9)$$

Hence

$$\frac{\phi_{m+4}(y) + \phi_{m+2}(y)}{\phi_{m+2}(y) + \phi_m(y)} = \frac{4(m+1)(m+2)}{(m+3)(m+4)} \cdot \frac{\psi_{m+2}(y)}{\psi_m(y)}$$

$$= \frac{y^2 - (m+2)^2}{(m+3)(m+4)}.$$

$$\therefore (m+3)(m+4)\phi_{m+4}(y) + \{(m+3)(m+4) + (m+2)^2 - y^2\}\phi_{m+2}(y) + \{(m+2)^2 - y^2\}\phi_m(y) = 0 \quad (10)$$

This is the required recurrence relation. Restoring s , x and η , it may be written

$$(2s+3)(2s+4)\eta_{s+2} + \{(2s+3)(2s+4) + (2s+2)^2 - 4x^2\}\eta_{s+1} + \{(2s+2)^2 - 4x^2\}\eta_s = 0 \quad (11)$$

The expressions for the polynomials corresponding to the first few values of s are given below:

$\eta_{1/2} = 2x$	$\eta_0 = 1$
$\eta_{3/2} = \frac{1}{3}(4x^3 - 7x)$	$\eta_1 = 2x^2 - 1$
$\eta_{5/2} = \frac{1}{60}(16x^5 - 120x^3 + 149x)$	$\eta_2 = \frac{1}{3}(2x^4 - 8x^2 + 3)$
$\eta_{7/2} = \frac{1}{2520}(64x^7 - 1232x^5 + 6076x^3 - 6483x)$	$\eta_3 = \frac{1}{45}(4x^6 - 50x^4 + 136x^2 - 45)$

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REFERENCE

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| 1. Madhava Rao, Thiruvengkatachar and Venkatachaliengar | <i>Proc. Roy. Soc., (A), 1946, 187.</i> |
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